Some new couple common fixed point theorems for a pair of commuting mappings involving quadratic terms in partially ordered complete metric spaces

Rakesh Tiwari and Savita Gupta[1]

[1].

Abstract

The purpose of this paper is to establish some coupled coincidence point for a pair of commuting mappings involving quadratic terms in partially ordered complete metric spaces. We also present a result on the existence and uniqueness of coupled common fixed points. We provide an example to validate our results.

1.Introduction

S. Banach [5] proved the famous and well known Banach contraction principle concerning the fixed point of contraction mappings defined on a complete metric space. This theorem has been generalized and extended by many authors see for ([1],[2],[9],[13],[15],[8]) in various ways. Recently, Ran and Reurings [16], Bhaskar and Lakshmikantham [6], Nieto and Lopez [13], Agarwal, El-Gebeily and O'Regan [15] and Lakshmikantham and Ciric [7] presented some new results for contractions in partially ordered metric spaces. There after, many authors obtained many coupled coincidence and coupled fixed point theorems in ordered metric spaces (see [1],[3],[11],[12],[13],[14] as examples). For a given partially ordered set, Bhaskar and Lakshmikantham [6] introduced the concept of coupled fixed point of a mapping. Later Lakshmikantham and Ciric [10] investigated some more coupled fixed point theorems in partially ordered sets. Very recently, Samet [17] extended the results of Bhaskar and Lakshmikantham [6] to mappings satisfying a generalized Meir-Keeler contractive condition.

2.Preliminaries

Let us recall the following definitions of coupled fixed point and mixed monotone properties of a mapping.

Definition 2.1.([6]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$. The mapping F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone non increasing in y, that is, for any $x, y \in X$, $x_1, x_2 \in X$, $x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$ and $y_1, y_2 \in X, y_1 \preceq y_2 \Rightarrow F(x, y_1) \succeq F(x, y_2)$. This definition

[1]Corresponding author

2000 Mathematics Subject Classification : Primary : 47H10, 54H25

Key Words and Phrases : Coupled fixed point, Partially ordered set, Mixed monotone property.

International Journal of Scientific & Engineering Research, Volume 7, Issue 10, October-2016
ISSN 2229-5518

coincides with the notion of a mixed monotone function on R_2 and represents the usual total order in R.

Definition 2.2.([6]). We call an element $(x, y) \in X \times X$ a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.
The concept of the mixed monotone property is generalized in [10].

Definition 2.3. ([10]). Let $(X, \preceq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. The mapping F is said to have the mixed g-monotone property if F is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument, that is, for any $x, y \in X$

$$x_1, x_2 \in X,\ g(x_1) \preceq g(x_2) \Rightarrow F(x_1, y) \preceq F(x_2, y) \quad (1)$$

and $$y_1, y_2 \in X,\ g(y_1) \preceq g(y_2) \Rightarrow F(x, y_1) \succeq F(x, y_2). \quad (2)$$

Clearly, if g is the identity mapping, then Definition 2.3 reduces to Definition 2.1.

Definition 2.4. [6]. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \longrightarrow X$ and $g : X \times X$ if $F(x, y) = g(x)$, and $F(y, x) = g(y)$.

Definition 2.5. Let (X,d) be a metric space and $F : X \times X \to X$ and $g : X \to X$ be mappings. We say F and g commute if $F(g(x), g(y)) = g(F(x, y))$ for all $x, y \in X$.

In this paper we proved, Some new couple common fixed point theorems for a pair of commuting mappings involving quadratic terms in partially ordered complete metric space.

3. Main result

The following theorems are is our main results.

Theorem 3.1. Let $(X, d, \preceq)$ be an ordered metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β, with $\alpha + \beta < 1$ such that

$$\begin{aligned} d^2(F(x,y),F(u,v)) \leq\ & \alpha\ min\{d(F(x,y),g(x))d(F(u,v),g(x)), d(F(u,v),g(x))d(F(x,y),g(u))\} \\ & +\beta\ min\{d(F(x,y),g(u))d(F(u,v),g(u)), d(F(u,v),g(x))d(F(x,y),g(u))\} \end{aligned} \quad (3)$$

for every $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \to g(X)$ and $g(X)$ is a complete subspace of X. Also, suppose that X satisfies the following properties:
(i) if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,
(ii) if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n. Then there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$, that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Suppose $x_0, y_0 \in X$ be such that $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$. Similarly we construct, $g(x_2) = F(x_1, y_1)$ and $g(y_2) = F(y_1, x_1)$. Continuing in this way we

International Journal of Scientific & Engineering Research, Volume 7, Issue 10, October-2016
ISSN 2229-5518

construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that,
$g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ for all $n \geq 0$. (4)
Now we prove that for all $n \geq 0$,

$$g(x_n) \preceq g(x_{n+1}) \tag{5}$$

and

$$g(y_n) \succeq g(y_{n+1}). \tag{6}$$

We shall use the method of mathematical induction. Let n = 0. Since $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$, in view of $g(x_1) = F(x_0, y_0)$ and $g(y_1) = F(y_0, x_0)$, we have $g(x_0) \preceq g(x_1)$ and $g(y_0) \succeq g(y_1)$, that is, (5) and (6) hold for n = 0. We presume that (5) and (6) hold for some $n > 0$. As F has the mixed g-monotone property and $g(x_n) \preceq g(x_{n+1})$, $g(y_n) \succeq g(y_{n+1})$, from (4), we get

$$g(x_{n+1}) = F(x_n, y_n) \preceq F(x_{n+1}, yn) \tag{7}$$

and

$$F(y_{n+1}, x_n) \succeq F(y_n, x_n) = g(y_{n+1}). \tag{8}$$

Also for the same reason we have $g(xn + 2) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n)$ and $F(y_{n+1}, x_n) \succeq F(y_{n+1}, x_{n+1}) = g(y_{n+2})$.
Then from (4) and (5), we obtain $g(x_{n+1}) \preceq g(x_{n+2})$ and $g(y_{n+1}) \succeq g(y_{n+2})$. Thus by the mathematical induction, we conclude that (5) and (6) hold for all $n = 0$.
We check easily that

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq ... \preceq g(x_{n+1}) \preceq ...$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq .... \succeq g(y_{n+1}) \succeq ....$$

Since

$$g(x_n) \succeq g(x_{n-1}) \text{ and } g(y_n) \preceq g(y_{n-1}),$$

Also for the same reason we have
$g(x_{n+2}) = F(x_{n+1}, y_{n+1}) \succeq F(x_{n+1}, y_n)$ and $F(y_{n+1}, x_n) \preceq F(y_{n+1}, x_{n+1}) = g(_{yn+2})$.
Then from (4) and (5), we obtain $g(x_{n+1}) \preceq g(x_{n+2})$ and $g(y_{n+1}) \succeq g(y_{n+2})$. Thus by the mathematical induction, we conclude that (5) and (6) hold for all $n \geq 0$. We check easily that

$$g(x_0) \preceq g(x_1) \preceq g(x_2) \preceq \ \preceq g(x_{n+1}) \preceq ....$$

and

$$g(y_0) \succeq g(y_1) \succeq g(y_2) \succeq \ \succeq g(y_{n+1}) \succeq ....$$

Since $g(x_n) \succeq g(x_{n-1})$ and $g(y_n) \preceq g(y_{n-1})$, from (3) and (4), we have

International Journal of Scientific & Engineering Research, Volume 7, Issue 10, October-2016
ISSN 2229-5518

$$\begin{aligned} d^2(g(x_{n+1}), g(x_n)) &= d^2(F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\leq \alpha\ min\{d(F(x_n, y_n), g(x_n))d(F(x_{n-1}, y_{n-1}), g(x_n)), \\ &\quad d(F(x_{n-1}, y_{n-1}), g(x_n))d(F(x_n, y_n), g(x_{n-1}))\} \\ &\quad +\beta\ min\{d(F(x_n, y_n), g(x_{n-1}))d(F(x_{n-1}, y_{n-1}), g(x_{n-1})), \\ &\quad d(F(x_n, y_n), g(x_{n-1}))d(F(x_{n-1}, y_{n-1}), g(x_n))\} \end{aligned}$$

or

$$d^2(g(x_{n+1}), g(x_n)) \leq \beta\ d^2(g(x_n), g(x_{n-1})). \tag{9}$$

Similarly, since $g(y_{n-1}) \succeq g(y_n)$ and $g(x_{n-1}) \preceq g(x_n)$, from (3) and (4), we have

$$d^2(g(y_n), g(y_{n+1})) \leq \alpha\ d^2(g(y_n), g(y_{n-1})). \tag{10}$$

From (9) and (10), we have

$$\begin{aligned} d^2(g(x_{n+1}), g(x_n)) + d^2(g(y_n), g(y_{n+1})) &\leq \beta\ d^2(g(x_n), g(x_{n-1})) + \alpha\ d^2(g(y_n), g(y_{n-1})) \\ &\leq (\alpha+\beta)d^2(g(x_n), g(x_{n-1})) + (\alpha+\beta)d^2(g(y_n), g(_{yn-1})) \\ &= (\alpha+\beta)[d^2(g(x_n), g(x_{n-1})) + d^2(g(y_n), g(y_{n-1}))]. \end{aligned}$$

Setting $\rho_n = d^2(g(x_{n+}1), g(xn)) + d^2(g(y_{n+1}), g(yn))$ and $\delta = \alpha + \beta$, we get the sequence $\{\rho_n\}$ is decreasing as

$$0 \leq \rho_n \leq \delta\rho_{n-1} \leq \delta\rho_{n-2} \leq .... \leq \delta^n\rho_0$$

This implies

$$lim_{n\to\infty}\rho_n = lim_{n\to\infty}[d^2(g(x_{n+1}), g(x_n)) + d^2(g(y_{n+1}), g(y_n))] = 0. \tag{11}$$

Thus,

$$lim_{n\to\infty}d^2(g(x_{n+1}), g(x_n)) = 0 \text{ and } lim_{n\to\infty}d^2(g(y_{n+1}), g(y_n)) = 0.$$

In what follows, we shall prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences.
For each $m \geq n$, we have

$$d^2(g(x_m), g(x_n)) \leq d^2(g(x_m), g(x_{m-1})) + d^2(g(x_{m-1}), g(x_{m-2})) + ... + d^2(g(x_{n+1}), g(x_n))$$

and

$$d^2(g(y_m), g(y_n)) \leq d^2(g(_ym), g(y_{m-1})) + d^2(g(y_{m1}), g(y_{m-2})) + ... + d^2(g(y_{n+1}), g(y_n)).$$

Therefore

$$\begin{aligned} d^2(g(x_m), g(x_n)) + d^2(g(ym), g(yn)) &\leq \rho_{m-1} + \rho_{m-2} + ... + \rho_n \\ &\leq (\delta^{m-1} + \delta^{m-2} + ... + \delta^n)\rho_0 \\ &\leq \frac{\delta^n}{1-\delta}\rho_0 \end{aligned} \tag{12}$$

International Journal of Scientific & Engineering Research, Volume 7, Issue 10, October-2016
ISSN 2229-5518

which implies that

$$lim_{n,m\to\infty}[d^2(g(x_m), g(x_n)) + d^2(g(y_m), g(y_n))] = 0.$$

This implies that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is a complete subspace of X, there exists $(x, y) \in X \times X$ such that $g(x_n) \to g(x)$ and $g(y_n) \to g(y)$. Since $\{g(x_n)\}$ is a nondecreasing sequence and $g(x_n) \to g(x)$ and as $\{g(y_n)\}$ is a nonincreasing sequence and $g(y_n) \to g(y)$, by assumption we have $g(x_n) \preceq g(x)$ and $g(y_n) \succeq g(y)$ for all n. Since

$$\begin{aligned} d^2(g(x_{n+1}), F(x, y)) &= d^2(F(x_n, y_n), F(x, y)) \\ &\leq \alpha\ min\{d(g(x_{n+1}), g(x_n))d(F(x, y), g(x_n)), d(F(x, y), g(x_n))d(g(x_{n+1}), g(x))\} \\ &\quad + \beta\ min\{d(g(x_{n+1}), g(x))d(F(x, y), g(x)), d(F(x, y), g(x_n))d(g(x_{n+1}), g(x))\}. \end{aligned}$$

Taking the limit as $n \to \infty$ we get $d^2(g(x), F(x, y)) = 0$. Hence $g(x) = F(x, y)$. Similarly, we can show that $g(y) = F(y, x)$. Thus we proved that F and g have a coupled coincidence point. This completes the proof.

Theorem 3.2. Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β with $\alpha + \beta < 1$ such that

$$\begin{aligned} d^2(F(x, y), F(u, v)) &= \alpha\ min\{d(F(x, y), g(x))d(F(u, v), g(x)), d(F(u, v), g(x))d(F(x, y), g(u))\} \\ &\quad + \beta\ min\{d(F(x, y), g(u))d(F(u, v), g(u)), d(F(x, y), \\ &\quad g(u))d(F(u, v), g(x))\} \end{aligned} \tag{13}$$

for all $(x, y), (u, v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$, g is continuous nondecreasing and commutes with F , and also suppose either

(i) F is continuous or
(ii) X has the following property:
 (a) if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,
 (b) if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n,

Then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

Proof. Following the proof of Theorem 3.1, we have two Cauchy sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that $\{gx_n\}$ is a nondecreasing sequence in X and $\{gy_n\}$ is a nonincreasing sequence in X. Since X is a complete metric space, there is $(x, y) \in X \times X$ such that $gx_n \to x$ and $gy_n \to y$. Since g is continuous, we have $g(gx_n) \to gx$ and $g(gy_n) \to gy$. First, suppose that F is continuous. Then $F(gx_n, gy_n) \to F(x, y)$ and $F(gy_n, gx_n) \to F(y, x)$. On other hand, we have $F(gx_n, gy_n) = gF(x_n, y_n) = g(gx_{n+1}) \to gx$ and $F(gy_n, gx_n) = gF(y_n, x_n) = g(gy_{n+1}) \to gy$. By uniqueness of limit, we get $gx = F(x, y)$ and $gy = F(y, x)$.

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ISSN 2229-5518

Now, suppose that (ii) holds. Since $g(x_n)$ is a nondecreasing sequence such that $g(x_n) \to x$, $g(y_n)$ is a nonincreasing sequence such that $g(y_n) \to y$, and g is a nondecreasing function, we get that $g(gx_n) \preceq gx$ and $g(gy_n) \succeq g(y)$ holds for all $n \in N$. By (13), we have

$$\begin{aligned} d^2(g(gx_{n+1}), F(x,y)) &= d^2(F(gx_n, gy_n), F(x,y)) \\ &\leq \alpha\ min\{d(ggx_{n+1}, ggx_n)d(F(x,y), ggx_n), d(F(x,y), ggx_n)d(ggx_{n+1}, gx)\} \\ &\quad +\beta\ min\{d(ggx_{n+1}, gx)d(F(x,y), gx), d(ggx_{n+1}, gx)d(F(x,y), ggx_n)\}. \end{aligned}$$

Letting $n \to \infty$, we get $d(g(x), F(x,y)) = 0$ and hence $g(x) = F(x,y)$. Similarly, we can show that $g(y) = F(y,x)$. Thus we proved that F and g have a coupled coincidence point.

Corollary 3.3. Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$. X be a mapping such that F has the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \preceq F(x_0, y_0)$ and $y_0 \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α, β with $\alpha + \beta < 1$ such that

$$\begin{aligned} d(F(x,y), F(u,v)) &\leq \alpha\ min\{d(F(x,y), x)d(F(u,v), x), d(F(u,v), x)d(F(x,y), u)\} \\ &\quad +\beta\ min\{d(F(x,y), u)d(F(u,v), u), d(F(x,y), u)d(F(u,v), x)\} \end{aligned} \tag{14}$$

for all $(x,y), (u,v) \in X \times X$ with $x \succeq u$ and $y \preceq v$ and also suppose either
(i) F is continuous or
(ii) X has the following property:
(a) if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,
(b) if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n,
then there exist $x, y \in X$ such that $F(x,y) = x$ and $F(y,x) = y$, that is, F has a coupled fixed point $(x,y) \in X \times X$.

Proof. In Theorem 3.2, if g = I, the identity mapping, then we have the result.

Corollary 3.4. Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X,d) is a complete metric space. Let $F : X \times X \to X$ and $g : X \to X$ be mappings such that F has the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $g(x_0) \preceq F(x_0, y_0)$ and $g(y_0) \succeq F(y_0, x_0)$. Suppose there exist non-negative real numbers α and β with $\alpha + \beta < 1$ such that

$$\begin{aligned} d^2(F(x,y), F(u,v)) &= (\alpha+\beta)\ min\ \{d(F(x,y), g(x))d(F(u,v), g(x)), d(F(u,v), g(x))d(F(x,y), g(u)), \\ &\quad d(F(x,y), g(u))d(F(u,v), g(u)), d(F(x,y), g(u))d(F(u,v), g(x))\} \end{aligned}$$

for all $(x,y), (u,v) \in X \times X$ with $g(x) \preceq g(u)$ and $g(y) \succeq g(v)$. Further suppose $F(X \times X) \subseteq g(X)$, g is continuous nondecreasing and commutes with F , and also suppose either

(i) F is continuous or
(ii) X has the following property:
(a) if a nondecreasing sequence $\{x_n\}$ in X converges to $x \in X$, then $x_n \preceq x$ for all n,

(b) if a nonincreasing sequence $\{y_n\}$ in X converges to $y \in X$, then $y_n \succeq y$ for all n, then there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$, that is, F and g have a coupled coincident point $(x, y) \in X \times X$.

Proof. From Theorem 3.2, since α and β are non-negative real numbers, we have

$$\begin{aligned}(\alpha+\beta)\ min\ \{&d(F(x,y),g(x))d(F(u,v),g(x)), d(F(u,v),g(x))d(F(x,y),g(u)),\\ &d(F(x,y),g(u))d(F(u,v),g(u)), d(F(x,y),g(u))d(F(u,v),g(x))\}\\ \leq \alpha\ min\ \{&d(F(x,y),g(x))d(F(u,v),g(x)), d(F(u,v),g(x))d(F(x,y),g(u))\}\\ +\beta\ min\ \{&d(F(x,y),g(u))d(F(u,v),g(u)), d(F(x,y),g(u))d(F(u,v),g(x))\}\end{aligned}$$

Now we will prove the existence and uniqueness theorem of a coupled common fixed point. That is, if $(X, \preceq)$ is a partially ordered set, then we endow the product space $X \times X$ with the following partial order: for $(x, y), (u, v) \in X \times X$, $(u, v) \preceq (x, y) \Leftrightarrow x \succeq u, y \preceq v$.

Theorem 3.5. For every $(x, y), (y^*, x^*) \in X \times X$ there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(y^*, x^*), F(y^*, x^*))$. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Proof. We know that from Theorem 3.1, the set of coupled coincidence points of F and g is non-empty. Suppose (x, y) and (y^*, x^*) are coupled coincidence points of F , that is, $g(x) = F(x, y)g(y) = F(y, x), g(x^*) = F(y^*, x^*)$ and $g(y^*) = F(y^*, x^*)$,
then
$g(x) = g(x^*)$ and $g(y) = g(y^*)$. (15)

We suppose that, there exists $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable with $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Put $u_0 = u, v_0 = v$, and taking $u_1, v_1 \in X$ so that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then, similarly we can proof of Theorem 3.1, we define sequences $\{g(u_n)\}, \{g(v_n)\}$

$g(u_{n+1}) = F(u_n, v_n)$ and $g(v_{n+1}) = F(v_n, u_n)$ for all n.

Now, set $x_0 = x, y_0 = y, x^*_0 = x^*, y^*_0 = y.$ and similarly, define the sequences $\{g(x_n)\}, \{g(y_n)\}$ and $g(x^*_n), g(y^*_n)$. Then it is prove that easily
$g(x_n) \to F(x, y)$, $g(y_n) \to F(y, x)$,$g(x^*_n) \to F(x^*, y^*)$,
and $g(y^*_n) \to F(y^*, x^*)$ for all $n \geq 1$. Since $(F(x, y), F(y, x)) = (g(x_1), g(y_1)) = (g(x), g(y))$ and $(F(u, v), F(v, u)) = (g(u_1), g(v_1))$ are comparable, then $g(x) \preceq g(u_1)$ and $g(y) \succeq g(v_1)$. It is show that easily $(g(x), g(y))$ and $(g(u_n), g(v_n))$ are comparable, that is, $g(x) \preceq g(u_n)$ and $g(y) \succeq g(v_n)$ for all $n \geq 1$. Then from (3), we have

$$\begin{aligned}d^2(g(x),g(u_{n+1})) = d^2(F(x,y),F(u_n,v_n)) \leq \alpha\ min\ \{&d(F(x,y),g(v_n))d(F(u,v),g(v_n)),\\ &d(F(x,y),g(u_n))d(F(v_n,u_n),g(u_n))\}\\ +\beta\ min\{d(F(x,y),g(u_n))d(F(u_n,v_n),g(u_n))\}, &d(F(x,y),g(u_n))d(F(u_n,v_n),g(u_n)).\end{aligned}$$

Since $F(x, y) = g(x)$, we have
$d(g(x), g(u_{n+1})) \leq \beta min d(g(x), g(u_n)), d(F(u_n, v_n), g(u_n))$.

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Hence
$d(g(x), g(u_{n+1})) \leq \beta d(g(x), g(u_n)).$ (16)

Now we again from (3), we have

$$\begin{aligned} d^2(g(v_{n+1}), g(y)) = d^2(F(v_n, u_n), F(y, x)) \leq\ & \alpha\ min\{d(F(v_n, u_n), g(v_n))d(F(x, y), g(v_n)), d(F(y, x), \\ & g(v_n))d(F(y, x), g(y))\} \\ & +\beta\ min\{d(F(v_n, u_n), g(y))d(F(y, x), g(y)), \\ & d(F(v_n, u_n), g(y))d(F(y, x), g(v_n))\}. \end{aligned}$$

Since $F(y, x) = g(y)$, we have $d(g(v_{n+1}), g(y)) \leq \alpha\ min\{d(F(v_n, u_n), g(v_n)), d(g(y), g(v_n))\}$.
Hence
$d(g(v_{n+1}), g(y)) \leq \beta d(g(v_n), g(y)).$ (17)

Then by (17) and (18), we have

$$\begin{aligned} d^2(g(x), g(u_{n+1})) + d^2(g(y), g(v_{n+1})) &\leq \beta d^2(g(x), g(u_n)) + \alpha d^2(g(v_n), g(y)) \\ &\leq (\alpha + \beta)[d^2(g(x), g(un)) + d^2(g(y), g(vn))] \\ &\leq (\alpha + \beta)^2[d^2(g(x), g(u_{n-1})) + d^2(g(y), g(v_{n-1}))] \\ &. \\ &. \\ &. \\ &\leq (\alpha + \beta)^{n+1}[d^2(g(x), g(u_0)) + d^2(g(y), g(v_0))]. \end{aligned}$$

Taking limit as $n \to \infty$,
we get $lim_{n\to\infty}[d(g(x), g(u_n)) + d(g(y), g(v_n))] = 0.$
It implies that
$lim_{n\to\infty} d(g(x), g(u_n)) = lim_{n\to\infty} d(g(y), g(v_n)) = 0.$ (18)
Similarly, we can show that
$lim_{n\to\infty} d(g(x^*), g(u_n)) = lim_{n\to\infty} d(g(y^*), g(v_n)) = 0.$ (19)

By the triangle inequality, (18) and (19),
$d(g(x), g(x^*)) \leq d(g(x), g(u_{n+1})) + d(g(x^*), g(u_{n+1})) \to 0$ as $n \to \infty$,
$d(g(y), g(y^*)) \leq d(g(y), g(v_{n+1})) + d(g(y^*), g(v_{n+1})) \to 0$ as $n \to \infty$,

we have $g(x) = g(x^*)$ and $g(y) = g(y^*)$. Thus we have (16). This implies that $(g(x), g(y)) = (g(x^*), g(y^*))$. Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$, by commutativity of F and g, we have

$g(g(x)) = g(F(x, y)) = F(g(x), g(y))$ and $g(g(y)) = g(F(y, x)) = F(g(y), g(x)).$ (20)
Denote $g(x) = z, g(y) = w$. Then from (21),
$g(z) = F(z, w)$ and $g(w) = F(w, z).$ (21)

That is (z,w) is a coupled coincidence point. Then from (21) with $x^* = z$ and $y^* = w$ it follows $g(z) = g(x)$ and $g(w) = g(y)$, that is,

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$g(z) = z$ and $g(w) = w$. (22)
From (21) and (22), $z = g(z) = F(z, w)$ and $w = g(w) = F(w, z)$. Therefore, (z,w) is a coupled common fixed point of F and g.
To prove the uniqueness, suppose that (p, q) is another coupled common fixed point. Then by (19) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$.

Corollary 3.6. For every $(x, y), (y^*, x^*) \in X \times X$ there exists $a(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Then F has a unique coupled fixed point, that is, there exist a unique $(x, y) \in X \times X$ such that $x = F(x, y)$ and $y = F(y, x)$.
Proof. In Theorem 3.3, if g = I, the identity mapping, then we have the result.

Theorem 3.7. From Theorem 3.1, if gx_0 and gy_0 are comparable then F and g have a coupled coincidence point (x,y) such that $gx = F(x, y) = F(y, x) = gy$.
Proof. By Theorem 3.1 we construct two sequences x_n and y_n in X such that $gx_n \to gx$ and $gy_n \to gy$, where (x,y) is a coincidence point of F and g. Suppose $gx_0 \preceq gy_0$, then it is an easy matter to show that $gx_n \preceq gy_n$ and for all $n \in N \cup 0$. Thus, by (3) we have

$$\begin{aligned} d^2(gx_n, gy_n) &= d^2(F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})) \\ &\leq \alpha\ min\ \{d(F(x_{n-1}, y_{n-1}), gx_{n-1})d(F(y_{n-1}, x_{n-1}), gx_{n-1}), \\ &\quad d(F(y_{n-1}, x_{n-1}), gx_{n-1})d(F(x_{n-1}, y_{n-1}), gy_{n-1})\} \\ &\quad +\beta\ min\ \{d(F(x_{n-1}, y_{n-1}), g_{y_{n-1}})d(F(y_{n-1}, x_{n-1}), gy_{n-1}), \\ &\quad d(F(x_{n-1}, y_{n-1}), g_{y_{n-1}})d(F(y_{n-1}, xy_{n-1}), gx_{n-1})\} \\ &= \alpha\ min\ \{d(gx_n, gx_{n-1})d(gy_n, gx_{n-1}), d(gy_n, gx_{n-1})d(gx_n, gy_{n-1})\} \\ &\quad +\beta\ min\ \{d(gx_n, gy_{n-1})d(gy_n, gy_{n-1}), d(gx_n, gy_{n-1})d(gy_n, gx_{n-1})\}. \end{aligned}$$

Letting the limit as $n \to \infty$, we get $d(gx, gy) = 0$. Hence $F(x, y) = gx = gy = F(y, x)$. A similar argument can be used if $gy_0 \preceq gx_0$.

Corollary 3.4. In addition to hypotheses of Theorem 3.1, if x_0 and y_0 are comparable then F has a coupled fixed point of the form (x, x).

Proof. From Theorem 3.7, if g = I, the identity mapping, then we have the result. We proof the Theorem 3.1 with the help of the following example.

Example 3.1. Suppose $X = [0, 1]$. Then $(X, \leq)$ is a partially ordered set with the natural ordering of real numbers. Suppose $d(x, y) = |x - y|$ for $x, y \in X$. Define $g : X \to X$ by $g(x) = x^2$ and $F : X \times X \to X$ by

$$F(x, y) = \begin{cases} \frac{x^2 - y^2}{10}, & if \quad x \geq y; \\ 0, & if \quad x < y; \end{cases}$$

Then
(1) (X, d) is a complete metric space.
(2) g(X) is complete.
(3) $F(X \times X) \subseteq g(X) = X$.
(4) X satisfies (i) and (ii) of Theorem 3.1.

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(5) F has the mixed g-monotone property.

(6) F and g satisfy

$$\begin{aligned} d^2(F(x,y),F(u,v)) &\leq \frac{1}{5}min\{d(F(x,y),g(x))d(F(u,v),g(x)),d(F(u,v),g(x))d(F(x,y),g(u))\} \\ &+\frac{1}{5}min\{d(F(x,y),g(u))d(F(u,v),g(u)),d(F(x,y),g(u))d(F(u,v),g(x))\} \end{aligned}$$

for all $gx \preceq gu$ and $gy \succeq gv$. Thus by Theorem 3.1, F and g have a coupled coincidence point. Moreover (0, 0) is a coupled fixed point of F .

Proof. The proofs of (1)-(5) are clear. The proof of (6) is divided into the following cases:
Case 1. If $x \geq y$ and $u < v$, then we have

$$\begin{aligned} d^2(F(x,y),F(u,v)) &= d^2(\frac{x^2-y^2}{10},0) = (\frac{x^2-y^2}{10})^2 \leq \frac{x^4}{100} \leq \frac{9x^4}{50} \\ &\leq \frac{1}{5}(\frac{9x^2}{10}+\frac{y^2}{10})^2 = \frac{1}{5}d^2(\frac{x^2-y^2}{10},x^2) \\ &\leq \frac{1}{5}\{d^2(\frac{x^2-y^2}{10},x^2),d^2(0,x^2)\} \\ &\leq \frac{1}{5}min\{d(\frac{x^2-y^2}{10},x^2)d(0,x^2),d(0,x^2)(\frac{x^2-y^2}{10},u^2)\} \\ &+\frac{1}{5}min\{d^2(\frac{x^2-y^2}{10},u^2)d(0,u^2),d(0,x^2)(\frac{x^2-y^2}{10},u^2)\}. \end{aligned}$$

Case 2. If $x < y$ and $u \geq v$, then

$$\begin{aligned} d^2(F(x,y),F(u,v)) &= d^2(\frac{u^2-v^2}{10},0) = (\frac{u^2-v^2}{10})^2 \leq \frac{u^4}{100} \leq \frac{9u^4}{50} \\ &\leq \frac{1}{5}(\frac{9u^2}{10}+\frac{v^2}{10})^2 = \frac{1}{5}d^2(\frac{u^2-v^2}{10},u^2) \\ &\leq \frac{1}{5}\{d^2(\frac{u^2-v^2}{10},u^2),d^2(0,u^2)\} \\ &\leq \frac{1}{5}min\{d(\frac{u^2-v^2}{10},x^2)d(0,x^2),d(0,u^2)(\frac{u^2-v^2}{10},x^2)\} \\ &+\frac{1}{5}min\{d^2(\frac{u^2-y^2}{10},u^2)d(0,u^2),d(0,u^2)(\frac{u^2-v^2}{10},x^2)\}. \end{aligned}$$

Case 3. If $x \leq y$ and $u \geq v$, then

$$\begin{aligned} d^2(F(x,y),F(u,v)) = d^2(0,0) = 0 &\leq \frac{1}{5}min\{d(0,x^2)d(0,x^2),d(0,x^2)d(0,u^2)\} \\ &+\frac{1}{5}min\{d(0,u^2)d(0,u^2),d(0,u^2)d(0,x^2)\} \\ &\leq \frac{1}{5}d(0,x^2)d(0,x^2)+\frac{1}{5}d(0,u^2)d(0,u^2). \end{aligned}$$

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Case 4. If $x \geq y$ and $u \geq v$, then $v \leq y \leq x \leq u$. Hence

$$\begin{aligned} d^2(F(x,y),F(u,v)) &= d^2(\frac{x^2-y^2}{10},\frac{u^2-v^2}{10}) \\ &= \frac{1}{100}|u^2-v^2-x^2+y^2|^2 \\ &= \frac{1}{100}|u^2-x^2+y^2-v^2|^2 \\ &= \frac{1}{100}u^4 \\ &\leq \frac{1}{5}min\{d^2(\frac{x^2-y^2}{10},u^2),d^2(\frac{u^2-v^2}{10},u^2)\} \\ &\leq \frac{1}{5}min\{d(\frac{x^2-y^2}{10},x^2)d(\frac{u^2-v^2}{10},x^2),d(\frac{u^2-v^2}{10},x^2)d(\frac{x^2-y^2}{10},u^2)\} \\ &\quad +\frac{1}{5}min\{d(\frac{x^2-y^2}{10},u^2)d(\frac{u^2-v^2}{10},u^2),d(\frac{x^2-y^2}{10},u^2)d(\frac{u^2-v^2}{10},x^2)\}. \end{aligned}$$

In all the above cases, inequality (3) of Theorem 3.1 is satisfied for $\alpha = \beta = \frac{1}{5}$. Hence by Theorem 3.1, $(0,0)$ is a unique coupled coincidence point. Indeed for $x > y$ we have $F(y,x) = 0$ and since $F(y,x) = g(y)$ we have $y = 0$. Then $F(x,0) = g(x)$ implies $x = 0$. The cases $x = y$ or $x < y$ are similar.

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Rakesh Tiwari
Department of Mathematics
Govt.V.Y.T.PG.Autonomous College
Durg (C.G.)491001
India.
e-mail: rakeshtiwari66@gmail.com

[1]Savita Gupta
Department of Mathematics
Shri Shankaracharya Institute of Technology and Management
Bhilai(C.G.)492001
India.
e-mail: savita.gupta17@gmail.com